

Mean-Field Games with Rough Common Noise

Erhan Bayraktar¹

Xihao He¹

Xiang Yu²

Fengyi Yuan³

¹Department of Mathematics, University of Michigan

²Department of Applied Mathematics, The Hong Kong Polytechnic University

³School of Science and Engineering, The Chinese University of Hong Kong, Shenzhen

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1. Motivation
2. Pathwise Formulation
3. Compactification Approach
4. Brownian Common Noise
5. Remarks

- In classical MFG,

$$\mu_t = \mathcal{L}(X_t).$$

- With common noise,

$$\mu_t = \mathcal{L}(X_t \mid \mathcal{F}_t^B).$$

- The analytical formulation leads to stochastic HJB–Fokker–Planck systems or stochastic master equations.
- Existing results often require monotonicity, convexity, or strong regularity.

Difficulty

Conditional laws, enlarged filtrations, and weak convergence do not fit directly into the classical compactification argument.

Rough common noise

- The common shock is modeled by a deterministic rough path $\mathbf{B} \in \widehat{\mathcal{C}}^{0,\alpha}$.
- The idiosyncratic noise remains an individual Brownian motion W .
- The framework includes enhanced Brownian motion as a special case.
- The common factor can be treated pathwise as an observed trajectory or an approximation limit.

Remark

The equilibrium problem is solved for a fixed path \mathbf{B} , and the randomization step is treated later.

- ① A new MFG framework with rough common noise on a canonical space of relaxed controls.
- ② Existence of a pathwise mean-field equilibrium under mild assumptions.
- ③ Rough martingale problems with the presence of relaxed controls.
- ④ Equivalence between RSDEs and rough martingale formulations.
- ⑤ A connection to weak and strong equilibria with Brownian common noise.

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Model

$$dX_t = b(t, X_t, \mu_t, \alpha_t) dt + \sigma(t, X_t, \mu_t) dW_t + \sigma^0(t, X_t, \mu_t) d\mathbf{B}_t.$$

- W is the idiosyncratic Brownian motion.
- B is the common rough path.
- The drift is controlled; the diffusion coefficients depend on the state and the measure flow.
- The objective is

$$J(\alpha; \mu) = \mathbb{E} \left[\int_0^T f(t, X_t, \mu_t, \alpha_t) dt + g(X_T, \mu_T) \right].$$

Rough path notation

Enhanced path and controlled path

Fix $\mathbf{B} = (B, \mathbb{B}) \in \widehat{\mathcal{C}}^{0,\alpha}$, with increment $\delta B_{s,t} = B_t - B_s$. A path Z is controlled by B if

$$\delta Z_{s,t} = Z'_s \delta B_{s,t} + R_{s,t}^Z, \quad R_{s,t}^Z = O(|t - s|^{\beta+\beta'}).$$

Rough integral

For a controlled integrand (Z, Z') ,

$$\int_0^t Z_s d\mathbf{B}_s := \lim_{|\pi| \rightarrow 0} \sum_{[u,v] \in \pi} (Z_u \delta B_{u,v} + Z'_u \mathbb{B}_{u,v}).$$

- If B is smooth, this reduces to the usual pathwise integral.
- If \mathbf{B} is the Itô-enhanced Brownian motion, the rough integral agrees with the Itô integral for adapted integrands.

Prototype

Friz–Hocquet–Lê study equations of the form

$$dX_t = b(t, X_t) dt + \sigma(t, X_t) dW_t + (f_t, f'_t)(X_t) d\mathbf{B}_t,$$

with fixed rough path \mathbf{B} , Brownian motion W , and a controlled vector field (f, f') .

- The rough term is handled pathwise, while the stochastic term keeps the usual Itô structure.
- Under Lipschitz and rough-regularity assumptions, there is a strong well-posedness theory and a priori estimates.
- The solution X is itself a stochastic controlled rough path, so compositions and rough integrals remain stable.
- Later works connect this framework to controlled rough martingale problems, pathwise control, and dynamic programming.

Role in this talk

This is the engine behind the term $\sigma^0(t, X_t, \mu_t) d\mathbf{B}_t$ and behind the controlled lift $(\hat{\sigma}^0, \hat{\sigma}')$ used later.

Canonical space

- State space: $\mathcal{X} = C([0, T]; \mathbb{R}^d)$.
- Idiosyncratic noise space: $\mathcal{W} = C([0, T]; \mathbb{R}^\ell)$.
- Relaxed control space: \mathcal{Q} , where $q(dt, du) = q_t(du)dt$ on $[0, T] \times U$.
- Canonical space:

$$\Omega = \mathcal{X} \times \mathcal{Q} \times \mathcal{W},$$

with coordinate processes (X, Λ, W) .

- For fixed \mathbf{B} and candidate flow μ , an admissible control is a probability measure P on Ω .

Remark

Relaxed controls provide compactness while keeping the control–noise coupling explicit.

Definition 2.5

A law $P \in \mathcal{P}(\Omega)$ belongs to $R(\mathbf{B}, \lambda, \mu)$ if:

- 1 $(X_0)_\# P = \lambda$, and X_0 is independent of (Λ, W) ;
- 2 if $(\tilde{\sigma}^0, \tilde{\sigma}')$ is given by Lemma 2.4 and

$$\hat{\sigma}_t^0 = \tilde{\sigma}_t^0(X_t), \quad \hat{\sigma}_t' = (\nabla \tilde{\sigma}_t^0 \tilde{\sigma}_t^0 + \tilde{\sigma}_t')(X_t),$$

then $(X, \hat{\sigma}^0)$ and $(\hat{\sigma}^0, \hat{\sigma}')$ are stochastic controlled rough paths.

Definition 2.5

③ for every test function $\phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^\ell)$,

$$M_t(\phi) = \phi(X_t, W_t) - \int_0^t \int_U \widehat{L}\phi(s, X_s, W_s, \mu_s, u) \Lambda_s(du) ds \\ - \int_0^t (T_s(\phi), T'_s(\phi)) d\mathbf{B} - \int_0^t \text{Tr}(\sigma^0 \sigma^{0,\top}(s, X_s, \mu_s) \nabla_{xx}^2 \phi(X_s, W_s)) ds$$

is a martingale, where

$$\widehat{L}\phi = \widehat{b}^\top \nabla \phi + \frac{1}{2} \text{Tr}(\widehat{a} \nabla^2 \phi), \quad \widehat{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \quad \widehat{a} = \begin{pmatrix} \sigma \sigma^\top & \sigma \\ \sigma^\top & I_\ell \end{pmatrix},$$

and

$$T_t(\phi) = \nabla_x \phi(X_t, W_t) \widetilde{\sigma}_t^0(X_t), \quad T'_t(\phi) = \nabla_{xx}^2 \phi(X_t, W_t) (\widetilde{\sigma}_t^0, \widetilde{\sigma}_t^0) + \nabla_x \phi(X_t, W_t) \widehat{\sigma}'_t.$$

Special cases: $\phi = \phi(w)$ gives Brownianity of W ; $\phi = \phi(x)$ gives the rough martingale problem for X .

Definition 2.8

A law $Q \in \mathcal{P}(Q \times W)$ is causal if, for every t ,

$$\mathcal{F}_t^\Lambda \perp\!\!\!\perp \mathcal{F}_T^W \mid \mathcal{F}_t^W.$$

- This allows external randomization.
- Brownianity of W is preserved under filtration enlargement.

Proposition 2.9

For any causal coupling Q , initial law λ , and admissible flow μ , there exists

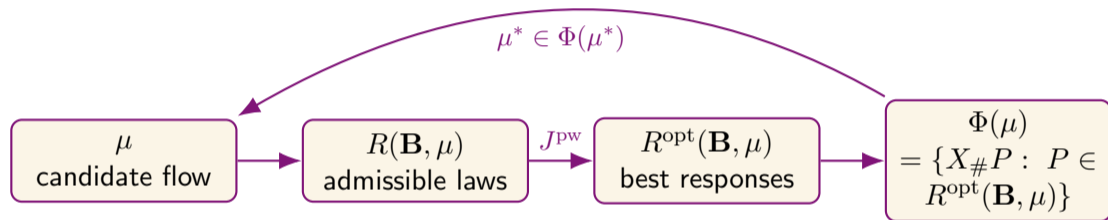
$$P \in R(\mathbf{B}, \lambda, \mu)$$

with

$$(\Lambda, W)_{\#} P = Q.$$

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Fixed-point argument



Remark

A fixed point of Φ gives a pathwise mean-field equilibrium.

Proposition 3.1

For admissible μ , the set $R(\mathbf{B}, \mu)$ is compact under weak convergence.

- Tightness comes from a priori rough-path estimates for X and the controlled lift of σ^0 .
- Closedness is delicate because the martingale formulation contains rough integrals and non-law-invariant norms.
- Appendix B supplies rough Itô formulas and quadratic-variation identities for $M^X(\phi)$.

Remark

Compactness is needed for Kakutani's fixed-point theorem.

Lemma 2.4

If μ is represented by a stochastic controlled rough path (Y, Y') , define

$$\tilde{\sigma}_t^0(x) = \sigma^0(t, x, \mu_t), \quad \tilde{\sigma}_t'(x) = \mathbb{E}[\partial_\mu \sigma^0(t, x, \mu_t)(Y_t) \cdot Y_t'].$$

Then $(\tilde{\sigma}^0, \tilde{\sigma}')$ is a controlled vector field with polynomial bounds.

- The measure flow must itself be represented by a controlled rough path.
- This is the ingredient that makes the rough integral in the state equation well defined.

Proposition 3.3

Every admissible law $P \in \mathcal{R}(\mathbf{B}, \mu)$ satisfies, P -a.s.,

$$X_t = X_0 + \int_0^t \int_U b(s, X_s, \mu_s, u) \Lambda_s(du) ds + \int_0^t \sigma(s, X_s, \mu_s) dW_s + \int_0^t (\tilde{\sigma}^0, \tilde{\sigma}') \circ (X, \tilde{\sigma}^0(X)) dB.$$

- The martingale formulation encodes the intended RSDE dynamics.
- This equivalence yields the key a priori estimate for the fixed-point argument.
- It is also used in the uniqueness and continuity steps.

Proposition 3.8

If $P_1, P_2 \in R(\mathbf{B}, \mu)$ and $(\Lambda, W)_{\#} P_1 = (\Lambda, W)_{\#} P_2$, then $P_1 = P_2$.

Lemma 3.9

The set-valued map $\mu \mapsto R^{\text{opt}}(\mathbf{B}, \mu)$ is upper hemicontinuous on $\mathcal{P}_{M,\varepsilon}$.

- Rough-path stability is formulated in rough norms.
- Kakutani is applied in weak topology.
- The local domain $\mathcal{P}_{M,\varepsilon}$ links these two levels.

Setup

Take $\mu_n \rightarrow \mu$ in $\mathcal{P}_{M,\varepsilon}$ and $P_n^* \in R^{\text{opt}}(\mathbf{B}, \mu_n)$ with $P_n^* \Rightarrow P^*$.

- 1 Put μ_n and P_n^* on common Skorokhod spaces; uniform local rough bounds on $(Y^n, Y^{n'})$ give uniform bounds on $(T^n(\phi), T^{n'}(\phi))$.
- 2 Continuity of rough integration passes the martingale problem to the limit, hence $P^* \in R(\mathbf{B}, \mu)$.
- 3 For any $P \in R(\mathbf{B}, \mu)$, Proposition 2.9 constructs $P^n \in R(\mathbf{B}, \mu_n)$ with $(\Lambda, W)_{\#} P^n = (\Lambda, W)_{\#} P$.
- 4 Any limit P° of P^n lies in $R(\mathbf{B}, \mu)$; Proposition 3.8 gives $P^\circ = P$, and passing to the limit in $J^{\text{pw}}(P^n; \mu_n) \geq J^{\text{pw}}(P_n^*; \mu_n)$ yields $P^* \in R^{\text{opt}}(\mathbf{B}, \mu)$.

Remark

- In Lacker (2015), continuity of admissible sets comes from strong SDE stability plus a Gronwall estimate.
- Here RSDE stability is formulated in rough-path norms, so the analogous route would require $\mu_n \rightarrow \mu$ in a rough-path topology, while our fixed-point argument uses weak topology.
- The paper therefore uses Proposition 3.8: fix the full control–noise law $(\Lambda, W)_{\#}P$, recover uniqueness from that joint law, and then pass to the limit in the optimality inequality.
- This is why the canonical definition must explicitly encode the idiosyncratic noise W .

Theorem 3.10

For any $m \geq 4$, there exists a pathwise mean-field equilibrium.

- Choose M, ε so that $\Phi(\mathcal{P}_{M,\varepsilon}) \subset \mathcal{P}_{M,\varepsilon}$.
- For each $\mu \in \mathcal{P}_{M,\varepsilon}$, the set $\Phi(\mu)$ is nonempty, convex, and compact.
- Lemma 3.9 gives upper hemicontinuity of Φ .
- Kakutani yields a fixed point μ^* and an equilibrium pair (P^*, μ^*) .

Remark

The equilibrium is obtained pathwise for each admissible common rough path \mathbf{B} .

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Randomization of the common noise

Brownian rough lift

Let

$$\mathbf{B} : \Omega^0 \ni \omega^0 \mapsto \left(\bar{B}^0(\omega^0), \int \bar{B}^0 \otimes d\bar{B}^0 \right) \in \widehat{\mathcal{C}}^{0,\alpha}.$$

- Section 3 solves the MFG pathwise for each fixed rough path B .
- Section 4 randomizes this input by pushing the Wiener measure forward through the rough-lift map \mathbf{B} .
- In the randomized setting, the relevant variable is the law of the full triple (X, Λ, W) , not only the marginal law of X .

Notation from Definition 4.1

For $\mu \in \mathcal{P}(\Omega)$,

$$\mu^x = X_{\#}\mu, \quad \mu_{\cdot, \wedge t} = (X_{\cdot, \wedge t}, \Lambda|_{[0, t] \times U}, W_{\cdot, \wedge t})_{\#}\mu.$$

Definition 4.1

$$\hat{\Omega}^0 = \hat{\mathcal{C}}^{0,\alpha} \times \mathcal{P}(\Omega), \quad \hat{\Omega} = \hat{\mathcal{C}}^{0,\alpha} \times \mathcal{P}(\Omega) \times \mathcal{P}(\Omega),$$

with canonical elements $(\hat{B}, \hat{\mu})$ and $(\hat{B}, \hat{\mu}, \hat{\nu})$.

Filtration generated by $\hat{\mu}$

$$\mathcal{F}_t^{\hat{\mu}} = \sigma\left(\mathbb{E}^{\hat{\mu}}[\phi(X_{\cdot \wedge t}, \Lambda|_{[0,t] \times U}, W_{\cdot \wedge t})] : \phi \in C_b\right).$$

- \hat{Q} on $\hat{\Omega}^0$ is causal if

$$\mathcal{F}_t^{\hat{\mu}} \perp\!\!\!\perp \mathcal{F}_T^{\hat{B}} \mid \mathcal{F}_t^{\hat{B}}, \quad t \in [0, T].$$

- \hat{P} on $\hat{\Omega}$ is causal if

$$\mathcal{F}_t^{\hat{\mu}, \hat{\nu}} \perp\!\!\!\perp \mathcal{F}_T^{\hat{B}} \mid \mathcal{F}_t^{\hat{B}}.$$

Causality is the law-level replacement of compatibility.

Weak equilibrium on the classical space

Definition 4.2

Set

$$T(\hat{Q}) = \mu(d\omega) \hat{Q}(dB, d\mu), \quad \bar{\Omega} = \Omega \times \hat{\Omega}^0, \quad \bar{P} := T(\hat{Q}).$$

Then $\hat{Q} \in \mathcal{P}^c(\hat{\Omega}^0)$ is a weak equilibrium if \bar{P} satisfies:

- 1 $(\bar{X}_0)_\# \bar{P} = \lambda$, and $\bar{X}_0, \bar{W}, (\bar{B}, \bar{\mu})$ are independent;
- 2 (\bar{W}, \bar{B}) are Brownian motions under the enlarged filtration;
- 3 the state equation

$$d\bar{X}_t = \int_U b(t, \bar{X}_t, \bar{\mu}_t^x, u) \bar{\Lambda}_t(du) dt + \sigma(t, \bar{X}_t, \bar{\mu}_t^x) d\bar{W}_t + \sigma^0(t, \bar{X}_t, \bar{\mu}_t^x) d\bar{B}_t;$$

- 4 no admissible nonanticipative deviation \bar{P}' has smaller cost.

Consistency is built into $T(\hat{Q})$:

$$\mathcal{L}_{\bar{P}}(\bar{X}, \bar{\Lambda}, \bar{W} \mid \bar{B}, \bar{\mu}) = \bar{\mu}.$$

Definition 4.3

For a fixed causal law \hat{Q} , define

$$\hat{\mathcal{R}}(\hat{Q}) = \left\{ \hat{P} \in \mathcal{P}^c(\hat{\Omega}) : (\hat{B}, \hat{\mu})_{\#} \hat{P} = \hat{Q}, \hat{P}(\hat{\nu} \in R(\hat{B}, \lambda, \hat{\mu})) = 1 \right\},$$

and

$$\hat{\mathcal{R}}^{\text{opt}}(\hat{Q}) = \left\{ \hat{P}^* \in \hat{\mathcal{R}}(\hat{Q}) : J(\hat{P}^*) = \inf_{\hat{P} \in \hat{\mathcal{R}}(\hat{Q})} J(\hat{P}) \right\}.$$

Diagonal coupling

$$\hat{P}_{\hat{Q}}^{\text{diag}}(dB, d\mu, d\nu) = \delta_{\mu}(d\nu) \hat{Q}(dB, d\mu).$$

The randomized best-response problem is now posed on laws of $(\hat{B}, \hat{\mu}, \hat{\nu})$.

Equivalence of the two formulations

If $(\bar{B}, \bar{\mu})_{\#} \bar{P} = \hat{Q}$ and

$$\hat{P} = \mathcal{L}^{\bar{P}}(\bar{B}, \bar{\mu}, \mathcal{L}(\bar{X}, \bar{\Lambda}, \bar{W} \mid \bar{B}, \bar{\mu})),$$

then

\bar{P} solves the Brownian state equation with $\bar{X}_0 \sim \lambda$

if and only if

$$\hat{P} \in \hat{\mathcal{R}}(\hat{Q}).$$

- Sufficiency converts pathwise admissibility into the classical SDE dynamics.
- Necessity reconstructs, from a Brownian weak solution, a pathwise admissible conditional law given $(\bar{B}, \bar{\mu})$.

Pathwise characterization of weak equilibrium

Proposition 4.5

$$\hat{Q} \text{ is a weak equilibrium} \iff \hat{P}_{\hat{Q}}^{\text{diag}} \in \hat{\mathcal{R}}^{\text{opt}}(\hat{Q}).$$

- The total cost rewrites as $J(\hat{P}) = \int_{\hat{\Omega}} J^{\text{pw}}(\mu, \nu) \hat{P}(dB, d\mu, d\nu)$, with $J^{\text{pw}}(\mu, \nu) = \int_{\Omega} \Gamma(x, q, \mu) \nu(d\omega)$.
- Sufficiency: causality of \hat{Q} gives the Brownian property of (\bar{W}, \bar{B}) under the enlarged filtration, and Lemma 4.4 yields the state equation.
- Necessity: for any $\hat{P} \in \hat{\mathcal{R}}(\hat{Q})$, replace $\hat{\nu}$ by $E[\hat{\nu} \mid \hat{B}, \hat{\mu}]$; convexity of the pathwise admissible set and linearity of J^{pw} imply optimality of the diagonal coupling.

Interpretation

The Brownian common-noise problem is recast as a law-level optimization over randomized pathwise admissible sets.

Strong equilibrium

Assumption 2

① b , σ , and σ^0 do not depend on the mean-field term μ ;

②

$$\int_{\Omega} (\Gamma(x, q, \mu) - \Gamma(x, q, \nu)) (\mu - \nu)(d\omega) \geq 0,$$

and equality implies $\mu^x = \nu^x$.

Proposition 4.6

Under Assumption 2, every weak equilibrium is in fact strong.

- Compare the diagonal coupling $\hat{P}_{\hat{Q}}^{\text{diag}}$ with the conditionally independent coupling $\hat{P}_{\hat{Q}}^{\text{ind}}$ given \hat{B} .
- Optimality plus strict monotonicity force $\hat{\mu}^x = \hat{\nu}^x$ almost surely.
- Under conditional independence, this common marginal is measurable with respect to $\sigma(\hat{B})$, so the consistency condition becomes strong.

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- The framework extends the compactification method beyond semimartingale common noise.
- It gives a pathwise interpretation of aggregate shocks compatible with rough-path approximation.
- It is natural for prescribed, observed, or stress-tested common trajectories.
- It suggests further problems: n -player limits, mean-field control, and numerical schemes based on smooth-path lifts.

- ① Mean-field games with rough common noise admit a pathwise formulation.
- ② Relaxed controls and rough martingale problems support a compactification argument.
- ③ The existence theorem follows from compactness, upper hemicontinuity, and Kakutani's theorem.
- ④ The same pathwise framework characterizes weak and strong equilibria in the Brownian common-noise setting.

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Thank you!