Dynamic portfolio selection for nonlinear law-dependent preferences

Zongxia Liang [1] Jianming Xia [2] Fengyi Yuan [1]

[1]Department of Mathematical Sciences, Tsinghua University
[2]RCSDS, NCMIS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences

January 21, 2024

Contents

Portfolio Selection: a type of stochastic control problem in finance

- Choose some portfolio π (the control/policy), obtain the final endowment X_T^π, which is random;
- Find the π^* such that $X_T^{\pi^*}$ is the "best";
- ...in what sense?
    - Mean-variance (MV) criterion:

$$\max \quad \mathcal{C}(X_T^\pi) := \left\{ \mathbb{E}[X_T^\pi] - \frac{\gamma}{2}\mathrm{Var}(X_T^\pi) \right\};$$

    - Expected Utility (EU) theory:

$$\max \quad \mathcal{C}(X_T^\pi) := \mathbb{E}[U(X_T^\pi)]$$

for some utility function U.

Dynamic portfolio selection

- Key sturcture that differs dynamic problems from static ones: the information (e.g., some filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$).
- At time t, when "evaluating" the portfolio π, the agent should look at $\mathcal{C}(X_T^\pi|\mathcal{F}_t)$ instead of naively $\mathcal{C}(X_T^\pi)$: the information is updated when time flows!
- Consistently optimizing $\mathcal{C}(X_T^\pi)$: *pre-committed* problems/solutions in literature; essentially reduced to static problems.
- For MV problems, if we maximize $\mathcal{C}(X_T^\pi|\mathcal{F}_t)$ at $t \in [0,T)$, the optimal portfolio should be (modified from Basak and Chabakauri 2010):

$$\pi_s^{t,*} = \frac{\mu}{\gamma\sigma^2}\frac{\xi_s}{\xi_t}e^{\left(\frac{\mu}{\sigma}\right)^2(T-s)}.$$

Expected Utility theory

- Unlike MV, EU is time-consistent because of the tower property: $\mathbb{E}\{\mathbb{E}[U(X_T)|\mathcal{F}_s]|\mathcal{F}_t\} = \mathbb{E}[U(X_t)|\mathcal{F}_t]$ if $s > t$.
- Can be seen as a *linear* functional of probability distribution: $\mathbb{E}[U(X_T)] = \langle U, \mathcal{L}(X_T)\rangle$.
- A behavioural economics perspective: linearity gives *independence axiom*[*].
- Time-consistency is good, but independence axiom is violated in empirical studies (Allais Paradox).
- $\longrightarrow$ preferences should be represented by *nonlinear* functional of probability distribution!

[*] For a preference $\succ$ defined on distributions, if $\mu \succ \mu'$, then for any μ'' and $p \in [0,1]$, $p\mu + (1-p)\mu'' \succ p\mu' + (1-p)\mu''$.

Modifications to EUT

Assume that for some (nonlinear) functional $g : \mathcal{P}_0 \to \mathbb{R}$, the preference is represented by g, i.e., for two terminal endowments X and X', $X \succ X'$ if and only if $g(\mathcal{L}(X)) \geq g(\mathcal{L}(X'))$.

- Rank dependent utility theory (RDUT) (Hu, Jin, and Zhou 2021):

$$g(\mathbb{P}_X) = \int_0^{\infty} w(\mathbb{P}(U(X) > y))\mathrm{d}y + \int_{-\infty}^{0} \left[w(\mathbb{P}(U(X) > y)) - 1\right] \mathrm{d}y.$$

- Betweenness preferences (relaxing independence to betweenness, Chew 1983):

$$g(\mathbb{P}_X) = \mathbb{E}[U(X, g(\mathbb{P}_X))].$$

Another interpretation: an endogenous reference point (disappointment aversion, e.t.c.).

Dynamic portfolio choice under nonlinear preferences

- Ideally, we aim to find a π to maximize $g(\mathbb{P}^t_{X^\pi_T})^*$ for any $t \in [0, T)$. Not possible unless g is linear.
- Without the existence of the global optimality, we aim to find a strategy that "*the investor has no incentive to deviate from*" (quoted from Basak and Chabakauri 2010):

$$g\left(\mathbb{P}^t_{X^{t,\pi'}_T}\right) - g\left(\mathbb{P}^t_{X^{\pi^*}_T}\right) \leq 0, \forall t, \pi'.$$

- Continuous time model: the deviation happens on an infinitesimal interval $[t, t+\varepsilon)$, let $\varepsilon \to 0$ and consider the "marginal incentives" of the agent.

---

* $\mathbb{P}^t_X := \mathcal{L}(X|\mathcal{F}_t)$.

The market

- An overall filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$, generated by a $(d+k)$-dimensional standard Brownian motion $W = (W^{\mathcal{S}}, W^{\mathcal{O}})$.
- $\mathcal{S}$: risks driving the stocks. $\mathcal{O}$: "orthogonal" risks, hence unhedgeable.
- There are d-stocks in the market, and their price processes $\{S_t^i, i = 1, 2, \cdots, d, t \in [0, T]\}$ follow the dynamics

$$\begin{cases} \mathrm{d}S_t^i = S_t^i[\theta^i(t)\mathrm{d}t + \sigma^i(t)\cdot \mathrm{d}W_t^{\mathcal{S}}], \\ S_0^i = s_0^i > 0, \end{cases}$$

- Allow θ, σ to be random (and adapted to the *overall* filtration $\mathbb{F}$, allowing stochastic factor models) and even non-Markovian.

The wealth dynamics

We consider two distinctive formulations of portfolio process π:

- When π models the proportion of the wealth invested into the stocks, the self-financing wealth process $\{X_t^\pi, 0 \leq t \leq T\}$ satisfies the following SDE

$$\begin{cases} \mathrm{d}X_t^\pi = X_t^\pi \pi_t^\dagger \theta(t)\mathrm{d}t + X_t^\pi \pi_t^\dagger \sigma(t) \cdot \mathrm{d}W_t^{\mathcal{S}}, \\ X_0^\pi = x_0. \end{cases}$$

- When π models the dollar amount invested into the stocks, the self-financing wealth process $\{X_t^\pi, 0 \leq t \leq T\}$ satisfies the following SDE

$$\begin{cases} \mathrm{d}X_t^\pi = \pi_t^\dagger \theta(t)\mathrm{d}t + \pi_t^\dagger \sigma(t) \cdot \mathrm{d}W_t^{\mathcal{S}}, \\ X_0^\pi = x_0. \end{cases}$$

Equilibrium porfolios

Recall: aim to find the portfolio from which the agent does not want to deviate on any *infinitesimal interval*.

- Deviation: for any $t \in [0,T)$, $\varepsilon \in (0, T-t)$, and $\varphi \in L^\infty(\mathcal{F}_t, \mathbb{R}^d)$, the perturbed strategy $\bar{\pi}^{t,\varepsilon,\varphi}$ is given by $\bar{\pi}^{t,\varepsilon,\varphi} := \bar{\pi} + \varphi \mathbb{1}_{[t,t+\varepsilon)}$. We will write $\bar{X} := X^{\bar{\pi}}$ and $\bar{X}^{t,\varepsilon,\varphi} := X^{\bar{\pi}^{t,\varepsilon,\varphi}}$ for simplicity.
- A portfolio $\bar{\pi}$ is said to be an equilibirum if

$$g\left(\mathbb{P}^t_{\bar{X}_T^{t,\varepsilon,\varphi}}\right) - g\left(\mathbb{P}^t_{\bar{X}_T}\right) \le o(\varepsilon), \forall t, \varphi$$

- The "*regret*" of implementing $\bar{\pi}$ is sublinear.

Formal definitions

Definition

(a) $\bar{\pi}$ is called a **Type-I** equilibrium strategy, if for any $t \in [0, T)$ and $\varphi \in L^\infty(\mathcal{F}_t, \mathbb{R}^d)$ such that $\bar{\pi}^{t,\varepsilon,\varphi} \in \mathcal{A}$ for all sufficiently small $\varepsilon > 0$, we have

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(g\left(\mathbb{P}^t_{\bar{X}_T^{t,\varepsilon,\varphi}}\right) - g\left(\mathbb{P}^t_{\bar{X}_T}\right) \right) \leq 0.$$

(b) $\bar{\pi}$ is called a **Type-II** equilibrium strategy if, for a.e. $t \in [0, T)$, any $\varphi \in L^\infty(\mathcal{F}_t, \mathbb{R}^d)$ such that $\bar{\pi}^{t,\varepsilon,\varphi} \in \mathcal{A}$ for all sufficiently small $\varepsilon > 0$, and any $\zeta \in L^\infty(\mathcal{F}_t)$ with $\zeta \geq 0$, we have

$$\limsup_{\varepsilon \to 0} \mathbb{E}\left[\frac{1}{\varepsilon} \left(g\left(\mathbb{P}^t_{\bar{X}_T^{t,\varepsilon,\varphi}}\right) - g\left(\mathbb{P}^t_{\bar{X}_T}\right) \right) \zeta \right] \leq 0.$$

Contents

Derivatives with respect to probability measures

Two existing notions in the (mean-field games/controls) literature:

- Lions' derivative: suppose g is defined on $\mathcal{P}_2$, and $\tilde{g}$ is its lifting to $L^2(\Omega)$. Then we define the (Lions') derivative of g at μ to be the function $\partial_\mu g(\mu, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\tilde{g}(X+Y) - \tilde{g}(X) = \mathbb{E}[\partial_\mu g(\mu, X) \cdot Y] + o(\|Y\|_{L^2}),$$

  where $X \sim \mu$.
- Linear derivatives: a function $\frac{\delta g}{\delta \mu}(\cdot, \cdot) : \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ such that

$$g(\nu) - g(\mu) = \int_0^1 \left\langle \frac{\delta g}{\delta \mu}(s\nu + (1-s)\mu, \cdot), \nu - \mu \right\rangle \mathrm{d}s.$$

Derivatives with respect to probability measures

Our definition (closer to linear derivative): a function $\nabla g(\cdot,\cdot) : \mathcal{P}_0 \times \mathbb{X} \to \mathbb{R}$ such that:

$$\frac{\mathrm{d}}{\mathrm{d}s} g(s\nu + (1-s)\mu) = \langle \nabla g(s\nu + (1-s)\mu, \cdot), \nu - \mu \rangle.$$

- A local version of linear derivative.
- Need less regularity and intergrability than Lions' derivative.
- More convenient for computation than the global definition of linear derivative.

Examples

- A benchmark example: $g(\mu) = \langle U, \mu \rangle$.

$$g(s\nu + (1-s)\mu) = s\langle U, \nu\rangle + (1-s)\langle U, \mu\rangle = \langle U, \nu\rangle + s\langle U, \nu - \mu\rangle.$$

$$\Longrightarrow \boxed{\nabla g(\mu, x) = U(x).}$$

Note that we do not need the smoothness of U.

Examples

- The simplest nonlinearity: $g(\mu) = F(\mathbb{E}^{\mu}[U(X)]) = F(\langle U, \mu \rangle)$.

$$g(s\nu + (1-s)\mu) = F\Big(\langle U, \nu \rangle + s \langle U, \nu - \mu \rangle \Big),$$

thus

$$\frac{\mathrm{d}}{\mathrm{d}s} g(s\nu + (1-s)\mu) = F'\Big(\langle U, \nu \rangle + s \langle U, \nu - \mu \rangle \Big) \langle U, \nu - \mu \rangle$$

$$= \Big\langle F'\Big(\langle U, s\nu + (1-s)\mu \rangle \Big) U, \nu - \mu \Big\rangle .$$

$$\Longrightarrow \boxed{\nabla g(\mu, x) = F'(\langle U, \mu \rangle) U(x).}$$

A more interesting example

- Implicit function: $\mathbb{E}[F(X - g(\mathbb{P}_X))] = 0$. Equivalently:

$$\int_{\mathbb{R}} F(x - g(\mu))\mu(\mathrm{d}x) = 0.$$

For $\mu_s = s\mu_1 + (1-s)\mu_0$, we have

$$s\int_{\mathbb{R}} F(x - g(\mu_s))\mu_1(\mathrm{d}x) + (1-s)\int_{\mathbb{R}} F(x - g(\mu_s))\mu_0(\mathrm{d}x) = 0$$

Taking derivatives (with respect to s):

$$\int_{\mathbb{R}} F(x - g(\mu_s))(\mu_1 - \mu_0)(\mathrm{d}x) - \left(\int_{\mathbb{R}} F'(x - g(\mu_s))\mu_s(\mathrm{d}x)\right)\dot{g}(\mu_s) = 0.$$

$$\Longrightarrow \boxed{\nabla g(\mu, x) = \frac{F(x - g(\mu))}{\int_{\mathbb{R}} F'(x - g(\mu))\mu(\mathrm{d}x)}}.$$

Contents

Standing assumptions

Assumption

(a) ∇g *exists and satisfies certain intergrability.*

(b) $\nabla g(\mu, \cdot)$ *is concave.*

(c) *There exist functions* $M_0 : \mathcal{P}_0 \times \mathcal{P}_0 \to \mathbb{R}$ *and* $M_1 : \mathcal{P}_0 \times \mathcal{P}_0 \to \mathbb{R}_+$ *such that*

$$g(\mu_1) - g(\mu_0) \leq M_1(\mu_1, \mu_0)\langle \nabla g(\mu_0, \cdot), \mu_1 - \mu_0 \rangle + M_0(\mu_1, \mu_0), \quad \forall \mu_0, \mu_1.$$

- (b) is a natural generalization of the concavity of utility function in the linear case. (c) is a relaxed concavity of g itself, which is trivial in linear case.
- Important future directions: remove (b) and/or (c).

Heuristic derivation of FOC

We only look at the dollar amount case:

$$\mathrm{d}X_t^{\pi} = \pi_t^{\dagger}\theta(t)\mathrm{d}t + \pi_t^{\dagger}\sigma(t)\cdot \mathrm{d}W_t^{\mathcal{S}}.$$

Let us also assume concavity for simplicity:

$$g(\mu_1) - g(\mu_0) \leq \langle \nabla g(\mu_0, \cdot), \mu_1 - \mu_0 \rangle.$$

With $\xi^t = \partial_x \partial_\mu g(\mathbb{P}^t_{\bar{X}_T}, \bar{X}_T)$, we have

$$g\left(\mathbb{P}^t_{\bar{X}_T^{t,\varepsilon,\varphi}}\right) - g\left(\mathbb{P}^t_{\bar{X}_T}\right) \leq \mathbb{E}_t[\xi^t(\bar{X}_T^{t,\varepsilon,\varphi} - \bar{X}_T)].$$

Recall that in the classical stochastic maximum principle: $\xi^t = U'(\bar{X}_T)$ (independent of t).

Heuristic derivation of FOC

With $p^t(s) = \mathbb{E}_s[\xi^t]$, we suppose*

$$\mathrm{d}p^t(s) = q^{t,\mathcal{S}}(s)\mathrm{d}W_s^{\mathcal{S}} + q^{t,\mathcal{O}}(s)\mathrm{d}W_s^{\mathcal{O}}.$$

By the equation of X,

$$\mathrm{d}(\bar{X}_s^{t,\varepsilon,\varphi} - \bar{X}_s) = \mathbb{1}_{[t,t+\varepsilon)}(s)\varphi^\dagger(\theta(s)\mathrm{d}s + \sigma(s)\mathrm{d}W_s^{\mathcal{S}}).$$

Apply Itô's formula to $p^t(s)(\bar{X}_s^{t,\varepsilon,\varphi} - \bar{X}_s)$, taking (conditional) expectation to cancel the Brownian motion term, we get

$$\mathbb{E}_t\xi^t[(\bar{X}_T^{t,\varepsilon,\varphi} - \bar{X}_T)] = \varphi^\dagger\mathbb{E}_t\left[\int_t^{t+\varepsilon}(\theta(s)p^t(s) + \sigma(s)q^{t,\mathcal{S}}(s))\mathrm{d}s\right].$$

Divided by ε and let $\varepsilon \to 0$, we expect:

$$\boxed{\theta(t)p^t(t) + \sigma(t)q^{t,\mathcal{S}}(t) = 0.}$$

* $q^{t,\mathcal{S}}$ and $q^{t,\mathcal{O}}$ come from martingale representation. Also, p^t and q^t are nothing but the adjoint processes in stochastic maximum principle.

Verifications theorem

For any random variable Y, denote by $Z^{Y,\mathcal{S}}$ and $Z^{Y,\mathcal{O}}$ the processes appearing in the martingale representation, or equivalently, the Z-term in the BSDE representation.

Theorem (The verification theorem: dollar amount strategies)

Suppose $\xi^t = \partial_x \nabla g(\mathbb{P}^t_{\bar{X}_T}, \bar{X}_T)$ *or* $\xi^t \in \partial_x \nabla g(\mathbb{P}^t_{\bar{X}_T}, \bar{X}_T)$ *in the non-smooth case. Under certain technical assumptions on* θ, σ, $\bar{\pi}$, $\mathbb{E}_s[\xi^t]$, $Z^{\xi^t,\mathcal{S}}$, M_0 *and* M_1, *if we have*

$$\boxed{\kappa(t)\mathbb{E}_t[\xi^t] + Z^{\xi^t,\mathcal{S}}(t) = 0}, \quad t \in [0, T],$$

then $\bar{\pi}$ *is a Type-I or Type-II equilibrium, depending on the technical assumptions we impose.*

Verification theorems

Theorem (The verification theorem: proportion strategies)

Suppose $\xi^t = \partial_x \nabla g(\mathbb{P}^t_{\bar{X}_T}, \bar{X}_T)$ *or* $\xi^t \in \partial_x \nabla g(\mathbb{P}^t_{\bar{X}_T}, \bar{X}_T)$ *in the non-smooth case. Under certain technical assumptions on* θ, σ, $\bar{\pi}$, ξ^t, $\mathbb{E}_s[\bar{X}_T \xi^t]$, $Z^{\bar{X}_T \xi^t, \mathcal{S}}$, M_0 *and* M_1, *if we have*

$$\boxed{(\kappa(t) - \sigma^\dagger(t)\bar{\pi}_t)\mathbb{E}_t[\bar{X}_T \xi^t] + Z^{\bar{X}_T \xi^t, \mathcal{S}}(t) = 0}, \quad t \in [0, T],$$

then $\bar{\pi}$ *is a Type-I or Type-II equilibrium, depending on the technical assumptions we impose.*

Contents

EU and MV

- EU example: Suppose we are in the dollar amount case, the market is complete and U is smooth (say, exponential utility). Then $\xi^t = U'(\bar{X}_T)$, and FOC$\Leftrightarrow Z_t^{U'(\bar{X}_T)} = -\kappa(t)\mathbb{E}_t[U'(\bar{X}_T)] \Leftrightarrow \mathbb{E}_s[U'(\bar{X}_T)] = \lambda Y_s$, with Y the (unique) pricing kernel: $\mathrm{d}Y_s = -\kappa(s)Y_s\mathrm{d}W_s$. Thus, $\bar{X}_T = U'^{(-1)}(\lambda Y_T)$.
- MV example: Because $g(\mathbb{P}_X) = \mathbb{E}[X] - \frac{\gamma}{2}\mathbb{E}[X^2] + \frac{\gamma}{2}(\mathbb{E}[X])^2$, we have $\partial_x \nabla g(\mathbb{P}_X, x) = 1 - \gamma x + \gamma\mathbb{E}[X]$, and

$$\xi^t = 1 - \gamma\bar{X}_T + \gamma\mathbb{E}_t[\bar{X}_T], \quad Z^{\xi^t,\mathcal{S}}(s) = -\gamma Z^{\bar{X}_T,\mathcal{S}}(s).$$

  Thus, FOC turns into a simple form $Z^{\bar{X}_T,\mathcal{S}}(t) = \frac{\kappa(t)}{\gamma}$. This reproduces results in Basak and Chabakauri 2010.

New examples: betweenness preferences

Let us consider a g determined by

$$\mathbb{E}\left[F\left(\frac{X}{g(\mathbb{P}_X)}\right)\right] = 0,$$

where $F : (0, \infty) \to \mathbb{R}$ is smooth, increasing, concave, and $F(1) = 0$.

- $g(\mathbb{P}_X)$ is in the scale of certainty equivalent of X, because $g(\delta_x) = x$;
- g has the positive homogeneity: $g(\mathbb{P}_{\lambda X}) = \lambda g(\mathbb{P}_X)$ for $\lambda > 0$;
- Taking $F = U_\gamma$, the CRRA utility function, we obtain the (certainty equivalent of) EU preference: $g(\mathbb{P}_X) = U_\gamma^{-1}(\mathbb{E}U_\gamma(X))$.

New examples: betweenness preferences

Proposition

g satisfies the standing assumptions, and

$$\nabla g(\mu, x) = F\left(\frac{x}{g(\mu)}\right) \frac{g(\mu)^2}{\int_0^\infty y F'\left(\frac{y}{g(\mu)}\right)\mu(\mathrm{d}y)}.$$

Idea (assume deterministic θ and σ for explicit solutions):

- Make an ansatz of the form of $\bar{\pi}$ (hence $\bar{X}_T$): $\bar{\pi}_t = (\sigma^\dagger(t))^{-1} a_t, \quad t \in [0, T]$, where a is unknown and deterministic.
- Express $g(\mathbb{P}^t_{\bar{X}_T})$ in terms of a with a (implicit but known) function H.
- Use the FOC to obtain an equation (ODE in our case) of a and solve it.

New examples: betweenness preferences

Use the form of derivative:

$$\bar{X}_T\xi^t = \frac{\bar{X}_T F'\left(\bar{X}_T/g(\mathbb{P}^t_{\bar{X}_T})\right)g(\mathbb{P}^t_{\bar{X}_T})}{\mathbb{E}_t\left[\bar{X}_T F'\left(\bar{X}_T/g(\mathbb{P}^t_{\bar{X}_T})\right)\right]}.$$

Use the FOC:

$$a_t = \kappa(t) + \frac{Z^{\bar{X}_T F'\left(\bar{X}_T/g(\mathbb{P}^t_{\bar{X}_T})\right)}(t)}{\mathbb{E}_t\left[\bar{X}_T F'\left(\bar{X}_T/g(\mathbb{P}^t_{\bar{X}_T})\right)\right]}.$$

New examples: betweenness preferences

Denote $A(t) = \int_t^T |a_s|^2 \mathrm{d}s$. Because

$$\bar{X}_T = \bar{X}_t e^{\int_t^T a_s^\dagger \kappa(s)\mathrm{d}s - \frac{1}{2}A(t)} R(t,T),$$

where $R(t,T) \sim e^{\sqrt{A(t)}\xi}$ and ξ is standard normal. Suppose we can solve from the definition of g a function H such that $H(y) = g(\mathbb{P}_{e^{\sqrt{y}\xi}})$, then from positive homogeneity we have

$$g(\mathbb{P}^t_{\bar{X}_T}) = \bar{X}_t e^{\int_t^T a_s^\dagger \kappa(s)\mathrm{d}s - \frac{1}{2}A(t)} H(A(t)).$$

$$\Longrightarrow \mathbb{E}_t\left[\bar{X}_T F'\left(\bar{X}_T / g(\mathbb{P}^t_{\bar{X}_T})\right)\right]$$
$$= \bar{X}_t e^{\int_t^T a_r^\dagger \kappa(r)\mathrm{d}r - \frac{1}{2}A(t)} \mathbb{E}\left[e^{\sqrt{A(t)}\xi} F'\left(e^{\sqrt{A(t)}\xi} / H(A(t))\right)\right].$$

New examples: betweenness preferences

Use Itô's formula to compute Z:

$$
\begin{aligned}
&Z^{\bar{X}_T F'\left(\bar{X}_T/g(\mathbb{P}^t_{\bar{X}_T})\right)}(t) \\
=&\bar{X}_t e^{\int_t^T a_r^\dagger \kappa(r)\mathrm{d}r - \frac{1}{2}A(t)} \mathbb{E}\left[e^{\sqrt{A(t)}\xi} \left(F'\left(\frac{e^{\sqrt{A(t)}\xi}}{H(A(t))} \right) \right.\right. \\
&\left.\left. + \frac{e^{2\sqrt{A(t)}\xi}}{H(A(t))} F''\left(\frac{e^{\sqrt{A(t)}\xi}}{H(A(t))} \right) \right) \right] a_t.
\end{aligned}
$$

FOC $\Longrightarrow$

$$
\boxed{a_t = \kappa(t) G(A(t))}, \quad t \in [0, T],
$$

with

$$
G(y) := \frac{H(y) \cdot \mathbb{E}\left[e^{\sqrt{y}\xi} F'\left(e^{\sqrt{y}\xi}/H(y) \right) \right]}{-\mathbb{E}\left[e^{2\sqrt{y}\xi} F''\left(e^{\sqrt{y}\xi}/H(y) \right) \right]}, \quad y \geq 0.
$$

New examples: betweenness preferences

We transform the equation of a to an ODE:

$$\begin{cases} A'(t) = -|\kappa(t)|^2 G(A(t))^2, & t \in [0,T), \\ A(T) = 0. \end{cases}$$

An autonomous ODE (after appropriate time change) with explicit solution:

$$A(t) = \mathcal{G}^{-1}\left(\int_t^T |\kappa(s)|^2 \mathrm{d}s\right).$$

Here, $\mathcal{G}(x) := \int_0^x \frac{1}{G(y)^2}\mathrm{d}y, \quad x \in [0,\infty]$ and we assume that $\mathcal{G}(\infty) > \int_0^T |\kappa(s)|^2 \mathrm{d}s$.

Conclusion of the first example

Proposition

If θ, σ are deterministic and g is the CRRA betweenness preference, then an equilibrium portfolio is given by

$$\bar{\pi}_t = (\sigma^\dagger(t))^{-1}\kappa(t)G\left(\mathcal{G}^{-1}\left(\int_t^T |\kappa(s)|^2 \mathrm{d}s\right)\right), \quad t \in [0,T),$$

provided that $\mathcal{G}(\infty) > \int_0^T |\kappa(s)|^2 \mathrm{d}s$.

- If $F = U_\gamma$ for $\gamma > 0$, $G(y) \equiv 1/\gamma \Longrightarrow \bar{\pi}$ is the Merton's solution;
- If $F = \int_0^\infty U_\gamma \mathbf{F}(\mathrm{d}\gamma)$ (mixed CRRA utility) for some compactly supported distribution $\mathbf{F}$, we can prove that $\mathcal{G}(\infty) = \infty$.

New examples: weighted utility

We consider a g, in which the distribution of terminal endowment X is weighted via a decreasing function of the realization of X:

$$g(\mathbb{P}_X) = \frac{\mathbb{E}[X^{1-\rho} \cdot X^\gamma]}{(1-\rho)\mathbb{E}[X^\gamma]}.$$

- To make g monotone and concave, we require $-1 < \gamma \leq 0$, $\gamma \leq \rho < \gamma + 1$.
- Decreasing weight function: put more weights on bad scenarios.
- Extensions to other types of weight functions are possible.

New examples: weighted utility

Proposition

Let g be given by weighted utility. Then standing assumptions are satisfied. Moreover,

$$\nabla g(\mu, x) = \frac{1}{1-\rho} \cdot \frac{x^{1-\rho+\gamma} \int_0^\infty x^\gamma \mu(\mathrm{d}x) - x^\gamma \int_0^\infty x^{1-\rho+\gamma} \mu(\mathrm{d}x)}{\left(\int_0^\infty x^\gamma \mu(\mathrm{d}x)\right)^2}.$$

Next question: how to transform FOC to something we can solve? Both powers of terminal endowments should be important:

$$Y_i(s) := \mathbb{E}_s[\bar{X}_T^{r_i}], \quad Z_i(s) := Z^{Y_i(T)}(s),$$

with $r_1 = \gamma$, $r_2 = 1 - \rho + \gamma$.

New examples: weighted utility

Use the form of derivative:

$$\bar{X}_T\xi^t = \bar{X}_T\partial_x\nabla g(\mathbb{P}^t_{\bar{X}_T}, \bar{X}_T) = \frac{r_2\bar{X}_T^{r_2}\mathbb{E}_t[\bar{X}^{r_1}] - r_1\bar{X}_T^{r_1}\mathbb{E}_t[\bar{X}_T^{r_2}]}{(1-\rho)(\mathbb{E}_t[\bar{X}_T^{r_1}])^2},$$

$$\Longrightarrow \mathbb{E}_s[\bar{X}_T\xi^t] = \frac{\lambda_2 Y_2(s)Y_1(t) + \lambda_1 Y_1(s)Y_2(t)}{Y_1(t)^2},$$

$$\Longrightarrow \mathbb{E}_t[\bar{X}_T\xi^t] = \frac{Y_2(t)}{Y_1(t)} \quad \text{and} \quad Z^{\bar{X}_T\xi^t}(t) = \frac{\lambda_2 Z_2(t)Y_1(t) + \lambda_1 Z_1(t)Y_2(t)}{Y_1(t)^2}.$$

Use FOC:

$$\sigma(t)\bar{\pi}_t = \kappa(t) + \lambda_2\hat{Z}_2(t) + \lambda_1\hat{Z}_1(t),$$

in which

$$\hat{Z}_i(s) = Z_i(s)/Y_i(s), \quad \lambda_1 = \frac{-\gamma}{1-\rho}, \quad \lambda_2 = \frac{1-\rho+\gamma}{1-\rho}.$$

New examples: weighted utility

How to determine $\hat{Z}_1$ and $\hat{Z}_2$? Consider $\hat{X} = \log X$, $\hat{Y}_i = \log Y_i$, we have the following FBSDE:

$$\begin{cases} \mathrm{d}\hat{Y}_i(s) = -\frac{1}{2}(\hat{Z}_i(s))^2 \mathrm{d}s + \hat{Z}_i(s)\mathrm{d}W_s, \quad i = 1, 2, \\ \hat{Y}_1(T) = r_1 \hat{X}_T, \hat{Y}_2(T) = r_2 \hat{X}_T, \end{cases}$$

$$\begin{cases} \mathrm{d}\hat{X}_s = (\bar{\pi}_s \theta(s) - \frac{1}{2}\sigma(s)^2 \bar{\pi}_s^2)\mathrm{d}s + \sigma(s)\bar{\pi}(s)\mathrm{d}W_s, \\ \hat{X}_0 = \log x_0. \end{cases}$$

We can now plug FOC into the forward equation, and get a BSDE (without a forward one) with respect to $\bar{Y}_i = \hat{Y}_i - r_i \hat{X}$, and solve $\hat{Z}_1$, $\hat{Z}_2$ from this (quadratic) BSDE.

New examples: weighted utility

Proposition

Let g be given by weighted utility. Then a Type-II equilibrium is given by

$$\sigma(t)\bar{\pi}_t = \frac{1}{\rho - 2\gamma}\kappa(t) + \frac{1}{\rho - 2\gamma}[\lambda_1 \bar{Z}_1(t) + \lambda_2 \bar{Z}_2(t)],$$

in which $\bar{Z}_1$ *and* $\bar{Z}_2$ *is a solution of*

$$\begin{cases} \mathrm{d}\bar{Y}_i(s) = -\frac{1}{2}\Big[\bar{Z}(s)^{\dagger}\mathbf{C}^i\bar{Z}(s) + \mathbf{c}_{i,i}\bar{Z}_i(s)\kappa(s) \\ \qquad\qquad + \mathbf{c}_{-i,i}\bar{Z}_{-i}\kappa(s) + \mathbf{b}_i|\kappa(s)|^2\Big]\,\mathrm{d}s \\ \qquad\qquad + \bar{Z}_i(s)\mathrm{d}W_s,\ i = 1, 2, \\ \bar{Y}_1(T) = \bar{Y}_2(T) = 0, \end{cases}$$

New examples: weighted utility

Well-posedness of the QBSDE?

- The system of QBSDE is more difficult than the one-dimensional equation.
- Our system of QBSDE is *fully quadratic* in the sense that in each equation, both components of Z are in quadratic orders. Existing results are not directly applicable.
- To ensure the existence and/or uniqueness, we need to impose certain *smallness* condition to validate contraction arguments. To this end, $\Theta := \int_0^T |\kappa(s)|^2 ds$ and $V(\Theta) := \sup_\tau \|\Theta - \mathbb{E}_\tau[\Theta]\|_\infty$, we suppose $V(\Theta)$ is small: the market price of risk is not "so random".

Lemma

For any sufficiently small $\varrho > 0$, there exists $V_0 > 0$ such that if $V(\Theta) < V_0$, then the QBSDE admits a unique solution $(\bar{Y}, \bar{Z}) \in (L^\infty(\mathbb{F}, \mathbb{R}))^2 \times (H^d_{\mathrm{BMO}})^2$ with $\|\bar{Z}\|_{\mathrm{BMO}} < \varrho$.

References I

Basak, Suleyman and Georgy Chabakauri (2010). "Dynamic Mean-Variance Asset Allocation". In: *The Review of Financial Studies* 23.8, pp. 2970–3016.

Chew, Soo Hong (1983). "A generalization of the quasilinear mean with applications to the measurement of income inequality and decision theory resolving the Allais Paradox". In: *Econometrica* 51.4, pp. 1065–1092.

Hu, Ying, Hanqing Jin, and Xun Yu Zhou (2021). "Consistent investment of sophisticated rank-dependent utility agents in continuous time". In: *Mathematical Finance* 31.3, pp. 1056–1095. arXiv: 2006.01979.

Thank you!