Mean-Field Games with Rough Common Noise

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Motivation

- In classical MFG,
$$\mu_t = \mathcal{L}(X_t).$$
- With common noise,
$$\mu_t = \mathcal{L}(X_t \mid \mathcal{F}_t^B).$$
- The analytical formulation leads to stochastic HJB–Fokker–Planck systems or stochastic master equations.
- Existing results often require monotonicity, convexity, or strong regularity.

Difficulty

Conditional laws, enlarged filtrations, and weak convergence do not fit directly into the classical compactification argument.

Rough common noise

- The common shock is modeled by a deterministic rough path $\mathbf{B} \in \widehat{\mathcal{C}}^{0,\alpha}$.
- The idiosyncratic noise remains an individual Brownian motion W.
- The framework includes enhanced Brownian motion as a special case.
- The common factor can be treated pathwise as an observed trajectory or an approximation limit.

Remark

The equilibrium problem is solved for a fixed path $\mathbf{B}$, and the randomization step is treated later.

Contributions

1. A new MFG framework with rough common noise on a canonical space of relaxed controls.
2. Existence of a pathwise mean-field equilibrium under mild assumptions.
3. Rough martingale problems with the presence of relaxed controls.
4. Equivalence between RSDEs and rough martingale formulations.
5. A connection to weak and strong equilibria with Brownian common noise.

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State equation

Model

$$dX_t = b(t, X_t, \mu_t, \alpha_t)\, dt + \sigma(t, X_t, \mu_t)\, dW_t + \sigma^0(t, X_t, \mu_t)\, d\mathbf{B}_t.$$

- W is the idiosyncratic Brownian motion.
- B is the common rough path.
- The drift is controlled; the diffusion coefficients depend on the state and the measure flow.
- The objective is

$$J(\alpha; \mu) = \mathbb{E}\Big[\int_0^T f(t, X_t, \mu_t, \alpha_t)\, dt + g(X_T, \mu_T) \Big].$$

Rough path notation

Enhanced path and controlled path

Fix $\mathbf{B} = (B, \mathbb{B}) \in \widehat{\mathcal{C}}^{0,\alpha}$, with increment $\delta B_{s,t} = B_t - B_s$. A path Z is controlled by B if

$$\delta Z_{s,t} = Z'_s\, \delta B_{s,t} + R^Z_{s,t}, \qquad R^Z_{s,t} = O\big(|t-s|^{\beta+\beta'}\big).$$

Rough integral

For a controlled integrand (Z, Z'),

$$\int_0^t Z_s\, d\mathbf{B}_s := \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} \big(Z_u\, \delta B_{u,v} + Z'_u\, \mathbb{B}_{u,v}\big).$$

- If B is smooth, this reduces to the usual pathwise integral.
- If $\mathbf{B}$ is the Itô-enhanced Brownian motion, the rough integral agrees with the Itô integral for adapted integrands.

Friz et al.: RSDE theory

Prototype

Friz–Hocquet–Lê study equations of the form

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t + (f_t, f_t')(X_t)\, d\mathbf{B}_t,$$

with fixed rough path $\mathbf{B}$, Brownian motion W, and a controlled vector field (f, f').

- The rough term is handled pathwise, while the stochastic term keeps the usual Itô structure.
- Under Lipschitz and rough-regularity assumptions, there is a strong well-posedness theory and a priori estimates.
- The solution X is itself a stochastic controlled rough path, so compositions and rough integrals remain stable.
- Later works connect this framework to controlled rough martingale problems, pathwise control, and dynamic programming.

Role in this talk

This is the engine behind the term $\sigma^0(t, X_t, \mu_t)\, d\mathbf{B}_t$ and behind the controlled lift $(\widehat{\sigma}^0, \widehat{\sigma}')$ used later.

Canonical space

- State space: $\mathcal{X} = C([0,T];\mathbb{R}^d)$.
- Idiosyncratic noise space: $\mathcal{W} = C([0,T];\mathbb{R}^\ell)$.
- Relaxed control space: $\mathcal{Q}$, where $q(dt,du) = q_t(du)dt$ on $[0,T] \times U$.
- Canonical space:

$$\Omega = \mathcal{X} \times \mathcal{Q} \times \mathcal{W},$$

  with coordinate processes (X, Λ, W).
- For fixed $\mathbf{B}$ and candidate flow μ, an admissible control is a probability measure P on Ω.

Remark

Relaxed controls provide compactness while keeping the control–noise coupling explicit.

Relaxed control

Definition 2.5

A law $P \in \mathcal{P}(\Omega)$ belongs to $R(\mathbf{B}, \lambda, \mu)$ if:

1. $(X_0)_{\#}P = \lambda$, and X_0 is independent of (Λ, W);
2. if $(\widetilde{\sigma}^0, \widetilde{\sigma}')$ is given by Lemma 2.4 and

$$\widehat{\sigma}_t^0 = \widetilde{\sigma}_t^0(X_t), \qquad \widehat{\sigma}_t' = \big(\nabla\widetilde{\sigma}_t^0\,\widetilde{\sigma}_t^0 + \widetilde{\sigma}_t'\big)(X_t),$$

then $(X, \widehat{\sigma}^0)$ and $(\widehat{\sigma}^0, \widehat{\sigma}')$ are stochastic controlled rough paths.

Relaxed control

Definition 2.5

3. for every test function $\phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^\ell)$,

$$M_t(\phi) = \phi(X_t, W_t) - \int_0^t \int_U \widehat{L}\phi(s, X_s, W_s, \mu_s, u)\, \Lambda_s(du)ds \\ - \int_0^t (T_s(\phi), T'_s(\phi))\, d\mathbf{B} - \int_0^t \mathrm{Tr}\big(\sigma^0 \sigma^{0,\top}(s, X_s, \mu_s) \nabla^2_{xx}\phi(X_s, W_s)\big)\, ds$$

is a martingale, where

$$\widehat{L}\phi = \widehat{b}^\top \nabla\phi + \tfrac{1}{2}\,\mathrm{Tr}\big(\widehat{a}\,\nabla^2\phi\big), \qquad \widehat{b} = \binom{b}{0}, \qquad \widehat{a} = \begin{pmatrix} \sigma\sigma^\top & \sigma \\ \sigma^\top & I_\ell \end{pmatrix},$$

and

$$T_t(\phi) = \nabla_x \phi(X_t, W_t)\widetilde{\sigma}^0_t(X_t), \qquad T'_t(\phi) = \nabla^2_{xx}\phi(X_t, W_t)(\widetilde{\sigma}^0_t, \widetilde{\sigma}^0_t) + \nabla_x\phi(X_t, W_t)\widehat{\sigma}'_t.$$

Special cases: $\phi = \phi(w)$ gives Brownianity of W; $\phi = \phi(x)$ gives the rough martingale problem for X.

Causal coupling

Definition 2.8

A law $Q \in \mathcal{P}(\mathcal{Q} \times \mathcal{W})$ is causal if, for every t,

$$\mathcal{F}_t^\Lambda \perp\!\!\!\perp \mathcal{F}_T^W \mid \mathcal{F}_t^W .$$

- This allows external randomization.
- Brownianity of W is preserved under filtration enlargement.

Proposition 2.9

For any causal coupling Q, initial law λ, and admissible flow μ, there exists

$$P \in R(\mathbf{B}, \lambda, \mu)$$

with

$$(\Lambda, W)_\# P = Q.$$

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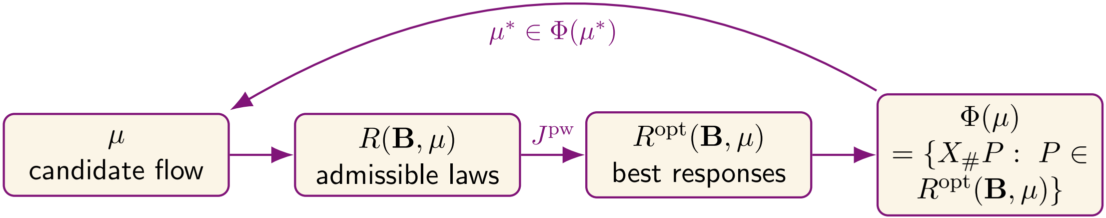
Fixed-point argument



Remark

A fixed point of Φ gives a pathwise mean-field equilibrium.

Compactness

Proposition 3.1

For admissible μ, the set $R(\mathbf{B}, \mu)$ is compact under weak convergence.

- Tightness comes from a priori rough-path estimates for X and the controlled lift of σ^0.
- Closedness is delicate because the martingale formulation contains rough integrals and non-law-invariant norms.
- Appendix B supplies rough Itô formulas and quadratic-variation identities for $M^X(\phi)$.

Remark

Compactness is needed for Kakutani's fixed-point theorem.

Controlled vector field

Lemma 2.4

If μ is represented by a stochastic controlled rough path (Y, Y'), define

$$\widetilde{\sigma}^0_t(x) = \sigma^0(t, x, \mu_t), \qquad \widetilde{\sigma}'_t(x) = \mathbb{E}\big[\partial_\mu \sigma^0(t, x, \mu_t)(Y_t) \cdot Y'_t\big].$$

Then $(\widetilde{\sigma}^0, \widetilde{\sigma}')$ is a controlled vector field with polynomial bounds.

- The measure flow must itself be represented by a controlled rough path.
- This is the ingredient that makes the rough integral in the state equation well defined.

RSDE and rough martingale problem

Proposition 3.3

Every admissible law $P \in R(\mathbf{B}, \mu)$ satisfies, P-a.s.,

$$X_t = X_0 + \int_0^t \int_U b(s, X_s, \mu_s, u)\, \Lambda_s(du)ds + \int_0^t \sigma(s, X_s, \mu_s)\, dW_s + \int_0^t (\widetilde{\sigma}^0, \widetilde{\sigma}') \circ (X, \widetilde{\sigma}^0(X))\, dB.$$

- The martingale formulation encodes the intended RSDE dynamics.
- This equivalence yields the key a priori estimate for the fixed-point argument.
- It is also used in the uniqueness and continuity steps.

Upper hemi-continuity

Proposition 3.8

If $P_1, P_2 \in R(\mathbf{B}, \mu)$ and $(\Lambda, W)_{\#}P_1 = (\Lambda, W)_{\#}P_2$, then $P_1 = P_2$.

Lemma 3.9

The set-valued map $\mu \mapsto R^{\mathrm{opt}}(\mathbf{B}, \mu)$ is upper hemicontinuous on $\mathcal{P}_{M,\varepsilon}$.

- Rough-path stability is formulated in rough norms.
- Kakutani is applied in weak topology.
- The local domain $\mathcal{P}_{M,\varepsilon}$ links these two levels.

Upper hemi-continuity: proof idea

Setup

Take $\mu_n \to \mu$ in $\mathcal{P}_{M,\varepsilon}$ and $P_n^* \in R^{\mathrm{opt}}(\mathbf{B}, \mu_n)$ with $P_n^* \Rightarrow P^*$.

1. Put μ_n and P_n^* on common Skorokhod spaces; uniform local rough bounds on $(Y^n, Y^{n\prime})$ give uniform bounds on $(T^n(\phi), T^{n\prime}(\phi))$.
2. Continuity of rough integration passes the martingale problem to the limit, hence $P^* \in R(\mathbf{B}, \mu)$.
3. For any $P \in R(\mathbf{B}, \mu)$, Proposition 2.9 constructs $P^n \in R(\mathbf{B}, \mu_n)$ with $(\Lambda, W)_\# P^n = (\Lambda, W)_\# P$.
4. Any limit P° of P^n lies in $R(\mathbf{B}, \mu)$; Proposition 3.8 gives $P^\circ = P$, and passing to the limit in $J^{\mathrm{pw}}(P^n; \mu_n) \geq J^{\mathrm{pw}}(P_n^*; \mu_n)$ yields $P^* \in R^{\mathrm{opt}}(\mathbf{B}, \mu)$.

Upper hemi-continuity: remark

Remark

- In Lacker (2015), continuity of admissible sets comes from strong SDE stability plus a Gronwall estimate.
- Here RSDE stability is formulated in rough-path norms, so the analogous route would require $\mu_n \to \mu$ in a rough-path topology, while our fixed-point argument uses weak topology.
- The paper therefore uses Proposition 3.8: fix the full control–noise law $(\Lambda, W)_{\#}P$, recover uniqueness from that joint law, and then pass to the limit in the optimality inequality.
- This is why the canonical definition must explicitly encode the idiosyncratic noise W.

Existence theorem

Theorem 3.10

For any $m \geq 4$, there exists a pathwise mean-field equilibrium.

- Choose M, ε so that $\Phi(\mathcal{P}_{M,\varepsilon}) \subset \mathcal{P}_{M,\varepsilon}$.
- For each $\mu \in \mathcal{P}_{M,\varepsilon}$, the set $\Phi(\mu)$ is nonempty, convex, and compact.
- Lemma 3.9 gives upper hemicontinuity of Φ.
- Kakutani yields a fixed point μ^* and an equilibrium pair (P^*, μ^*).

Remark

The equilibrium is obtained pathwise for each admissible common rough path $\mathbf{B}$.

Contents

Randomization of the common noise

Brownian rough lift

Let

$$\mathbf{B} : \Omega^0 \ni \omega^0 \mapsto \left(\bar{B}^0(\omega^0), \int \bar{B}^0 \otimes d\bar{B}^0 \right) \in \widehat{\mathcal{C}}^{0,\alpha}.$$

- Section 3 solves the MFG pathwise for each fixed rough path B.
- Section 4 randomizes this input by pushing the Wiener measure forward through the rough-lift map $\mathbf{B}$.
- In the randomized setting, the relevant variable is the law of the full triple (X, Λ, W), not only the marginal law of X.

Notation from Definition 4.1

For $\mu \in \mathcal{P}(\Omega)$,

$$\mu^x = X_{\#}\mu, \qquad \mu_{\cdot\wedge t} = (X_{\cdot\wedge t}, \Lambda|_{[0,t]\times U}, W_{\cdot\wedge t})_{\#}\mu.$$

Extended canonical spaces and causality

Definition 4.1

$$\hat{\Omega}^0 = \widehat{\mathcal{C}}^{0,\alpha} \times \mathcal{P}(\Omega), \qquad \hat{\Omega} = \widehat{\mathcal{C}}^{0,\alpha} \times \mathcal{P}(\Omega) \times \mathcal{P}(\Omega),$$

with canonical elements $(\hat{B}, \hat{\mu})$ and $(\hat{B}, \hat{\mu}, \hat{\nu})$.

Filtration generated by $\hat{\mu}$

$$\mathcal{F}_t^{\hat{\mu}} = \sigma\Big(\mathbb{E}^{\hat{\mu}}[\phi(X_{\cdot\wedge t}, \Lambda|_{[0,t]\times U}, W_{\cdot\wedge t})] : \phi \in C_b\Big).$$

- $\hat{Q}$ on $\hat{\Omega}^0$ is causal if

$$\mathcal{F}_t^{\hat{\mu}} \perp\!\!\!\perp \mathcal{F}_T^{\hat{B}} \mid \mathcal{F}_t^{\hat{B}}, \qquad t \in [0, T].$$

- $\hat{P}$ on $\hat{\Omega}$ is causal if

$$\mathcal{F}_t^{\hat{\mu},\hat{\nu}} \perp\!\!\!\perp \mathcal{F}_T^{\hat{B}} \mid \mathcal{F}_t^{\hat{B}}.$$

Causality is the law-level replacement of compatibility.

Weak equilibrium on the classical space

Definition 4.2

Set

$$T(\hat{Q}) = \mu(d\omega)\,\hat{Q}(dB, d\mu), \qquad \bar{\Omega} = \Omega \times \hat{\Omega}^0, \qquad \bar{P} := T(\hat{Q}).$$

Then $\hat{Q} \in \mathcal{P}^c(\hat{\Omega}^0)$ is a weak equilibrium if $\bar{P}$ satisfies:

1. $(\bar{X}_0)_{\#}\bar{P} = \lambda$, and $\bar{X}_0$, $\bar{W}$, $(\bar{B}, \bar{\mu})$ are independent;
2. $(\bar{W}, \bar{B})$ are Brownian motions under the enlarged filtration;
3. the state equation

$$d\bar{X}_t = \int_U b(t, \bar{X}_t, \bar{\mu}_t^x, u)\,\bar{\Lambda}_t(du)\,dt + \sigma(t, \bar{X}_t, \bar{\mu}_t^x)\,d\bar{W}_t + \sigma^0(t, \bar{X}_t, \bar{\mu}_t^x)\,d\bar{B}_t;$$

4. no admissible nonanticipative deviation $\bar{P}'$ has smaller cost.

Consistency is built into $T(\hat{Q})$:

$$\mathcal{L}_{\bar{P}}(\bar{X}, \bar{\Lambda}, \bar{W} \mid \bar{B}, \bar{\mu}) = \bar{\mu}.$$

Randomized admissible laws

Definition 4.3

For a fixed causal law $\hat{Q}$, define

$$\hat{\mathcal{R}}(\hat{Q}) = \Big\{\hat{P} \in \mathcal{P}^c(\hat{\Omega}) : (\hat{B}, \hat{\mu})_{\#}\hat{P} = \hat{Q},\ \hat{P}(\hat{\nu} \in R(\hat{B}, \lambda, \hat{\mu})) = 1\Big\},$$

and

$$\hat{\mathcal{R}}^{\mathrm{opt}}(\hat{Q}) = \Big\{\hat{P}^* \in \hat{\mathcal{R}}(\hat{Q}) : J(\hat{P}^*) = \inf_{\hat{P} \in \hat{\mathcal{R}}(\hat{Q})} J(\hat{P})\Big\}.$$

Diagonal coupling

$$\hat{P}^{\mathrm{diag}}_{\hat{Q}}(dB, d\mu, d\nu) = \delta_\mu(d\nu)\, \hat{Q}(dB, d\mu).$$

The randomized best-response problem is now posed on laws of $(\hat{B}, \hat{\mu}, \hat{\nu})$.

Lemma 4.4

Equivalence of the two formulations

If $(\bar{B}, \bar{\mu})_{\#}\bar{P} = \hat{Q}$ and

$$\hat{P} = \mathcal{L}^{\bar{P}}\big(\bar{B}, \bar{\mu}, \mathcal{L}(\bar{X}, \bar{\Lambda}, \bar{W} \mid \bar{B}, \bar{\mu})\big),$$

then

$$\bar{P} \text{ solves the Brownian state equation with } \bar{X}_0 \sim \lambda$$

if and only if

$$\hat{P} \in \hat{\mathcal{R}}(\hat{Q}).$$

- Sufficiency converts pathwise admissibility into the classical SDE dynamics.
- Necessity reconstructs, from a Brownian weak solution, a pathwise admissible conditional law given $(\bar{B}, \bar{\mu})$.

Pathwise characterization of weak equilibrium

Proposition 4.5

$$\hat{Q} \text{ is a weak equilibrium} \quad \Longleftrightarrow \quad \hat{P}^{\mathrm{diag}}_{\hat{Q}} \in \hat{\mathcal{R}}^{\mathrm{opt}}(\hat{Q}).$$

- The total cost rewrites as $J(\hat{P}) = \int_{\hat{\Omega}} J^{\mathrm{pw}}(\mu,\nu)\, \hat{P}(dB, d\mu, d\nu)$, with $J^{\mathrm{pw}}(\mu,\nu) = \int_{\Omega} \Gamma(x,q,\mu)\, \nu(d\omega)$.
- Sufficiency: causality of $\hat{Q}$ gives the Brownian property of $(\bar{W}, \bar{B})$ under the enlarged filtration, and Lemma 4.4 yields the state equation.
- Necessity: for any $\hat{P} \in \hat{\mathcal{R}}(\hat{Q})$, replace $\hat{\nu}$ by $E[\hat{\nu} \mid \hat{B}, \hat{\mu}]$; convexity of the pathwise admissible set and linearity of J^{pw} imply optimality of the diagonal coupling.

Interpretation

The Brownian common-noise problem is recast as a law-level optimization over randomized pathwise admissible sets.

Strong equilibrium

Assumption 2

1. b, σ, and σ^0 do not depend on the mean-field term μ;
2. $$\int_\Omega \left(\Gamma(x,q,\mu) - \Gamma(x,q,\nu)\right)(\mu-\nu)(d\omega) \geq 0,$$
   and equality implies $\mu^x = \nu^x$.

Proposition 4.6

Under Assumption 2, every weak equilibrium is in fact strong.

- Compare the diagonal coupling $\hat{P}^{\mathrm{diag}}_{\hat{Q}}$ with the conditionally independent coupling $\hat{P}^{\mathrm{ind}}_{\hat{Q}}$ given $\hat{B}$.
- Optimality plus strict monotonicity force $\hat{\mu}^x = \hat{\nu}^x$ almost surely.
- Under conditional independence, this common marginal is measurable with respect to $\sigma(\hat{B})$, so the consistency condition becomes strong.

Contents

Remarks

- The framework extends the compactification method beyond semimartingale common noise.
- It gives a pathwise interpretation of aggregate shocks compatible with rough-path approximation.
- It is natural for prescribed, observed, or stress-tested common trajectories.
- It suggests further problems: n-player limits, mean-field control, and numerical schemes based on smooth-path lifts.

Conclusion

1. Mean-field games with rough common noise admit a pathwise formulation.
2. Relaxed controls and rough martingale problems support a compactification argument.
3. The existence theorem follows from compactness, upper hemicontinuity, and Kakutani's theorem.
4. The same pathwise framework characterizes weak and strong equilibria in the Brownian common-noise setting.

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Thank you!